Comment on Shaw's refutation of the ρ bootstrap

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Gordon Shaw has argued from duality that exchange forces contribute a null term to the $\pi\pi$ P wave, hence that the traditional ρ bootstrap must fail in any model consistent with duality. Shaw's argument rests, however, on use of an infinite series in a region where it diverges. The contribution of exchange forces is computed here for the Veneziano model, and is shown to be nonzero. Although Shaw's argument is not valid, N/D calculations nevertheless indicate that exchange forces in the $\pi\pi$ channel are too weak to generate the ρ.

Shaw has argued that exchange forces in the $\pi\pi$ channel are incapable of generating the ρ resonance if the left cut of the P wave is consistent with duality and ρ-f_0 exchange degeneracy.[1] While I share Shaw's skepticism that the ρ is bound in the $\pi\pi$ channel,[2,3] Shaw's argument is flawed, and deserves comment.

Consider the standard decomposition of the $\pi\pi$ P wave $A^{(1)1}$ into a sum of terms A_L and A_R resulting from the left and right cuts, respectively:

$$A^{(1)1}(s)=\frac{s-4}{\pi}\left(\int_{-\infty}^{0}ds'+\int_{4}^{\infty}ds'\right)\frac{\operatorname{Im}A^{(1)1}(s')}{(s'-4)(s'-s)}$$

$$\equiv A_L(s)+A_R(s)\,,$$

where s denotes the center-of-mass energy squared, and $m_\pi=\hbar=c=1$. The term A_L embodies the exchange forces, and Shaw argued that A_L is zero.

Let A^I denote the $\pi\pi$ amplitude with isospin "I" in the s (direct) channel. In dual resonance models, it is customary to express the three A^I in terms of a single function $F(x,y)=F(y,x)$. A^1 is then given by

$$A^1(s,\cos\theta)=F(s,u)-F(s,t)\,, \tag{1}$$

where t and u denote the usual Mandelstam variables:

$$t\equiv-\tfrac{1}{2}(s-4)(1-\cos\theta)\,,$$

$$u\equiv-\tfrac{1}{2}(s-4)(1+\cos\theta)\,.$$

In the single-term Veneziano model, F is given by[4]

$$F(x,y)=\beta\,\frac{\Gamma(1-\alpha(x))\,\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))}\,, \tag{2}$$

where β denotes a normalization constant, and $\alpha(x)=a+bx$ denotes the exchange-degenerate ρ-f_0 Regge trajectory.

In its most explicit form, Shaw's argument proceeds as follows. The $F(x,y)$ of Eq. (2) can be expressed as[5]

$$F(x,y)=\beta\sum_{K=1}^{\infty}\frac{T_K(\alpha(y))}{(K-1)!}\,\frac{1}{\alpha(x)-K}\,, \tag{3}$$

where

$$T_K(\xi)\equiv\xi(\xi+1)(\xi+2)\cdots[\xi+(K-1)]$$

denotes the Kth-order Pochhammer polynomial. The series (3) converges for $\operatorname{Re}[\alpha(y)]<0$.

Shaw notes that Eq. (3) can be used to expand $F(s,t)$ and $F(s,u)$ as series of s-channel poles, hence it can be used to expand A^1 as a series of s-channel poles:

$$A^1(s,\cos\theta)=\beta\sum_{K=1}^{\infty}\frac{T_K(\alpha(u))-T_K(\alpha(t))}{(K-1)!}\,\frac{1}{\alpha(s)-K}\,. \tag{4}$$

If (as assumed by Shaw) Eq. (4) were valid throughout the s-channel physical region, it would follow that $\operatorname{Re}[A^1]$ is dual to direct-channel resonances, hence that $A_L=0$.

It is readily seen that Shaw's argument is not valid, because the series (4) *diverges* when $\alpha(t)>0$ and/or $\alpha(u)>0$.[5] Since $\alpha(0)>0$ for the ρ-f_0 trajectory, the series (4) cannot be used to express A^1 near the forward ($t=0$) or backward ($u=0$) directions, where A^1 is largest. [In fact, Eq. (4) is only valid for t and u such that A^1 tends asymptotically to zero.] Thus partial waves of A^1 *cannot* be expressed as projections of this series containing only direct-channel poles.

A series representation for A^1 which is valid throughout the s-channel physical region may be obtained from[5]

$$F(x,y)=\beta\sum_{K=1}^{\infty}\frac{(-1)^K}{(K-1)!}\,T_K(1-\alpha(x)-\alpha(y))\left(\frac{1}{\alpha(x)-K}+\frac{1}{\alpha(y)-K}\right), \tag{5}$$

which converges when $\mathrm{Re}(\alpha(x)+\alpha(y))>0$. Equations (1) and (5) yield

$$A^1(s,\cos\theta)=\beta\sum_{K=1}^{\infty}\frac{(-1)^K}{(K-1)!}\left[T_K(1-\alpha(s)-\alpha(u))\left(\frac{1}{\alpha(s)-K}+\frac{1}{\alpha(u)-K}\right)-T_K(1-\alpha(s)-\alpha(t))\left(\frac{1}{\alpha(s)-K}+\frac{1}{\alpha(t)-K}\right)\right], \quad (6)$$

which converges for $\mathrm{Re}(s)>-2a/b$ when $|\cos\theta|\leq 1$. Hence Eq. (6) is valid throughout the s-channel physical region, but contains crossed-channel poles as well as direct-channel poles. The crossed-channel contributions do not cancel each other completely,[6] for I find by direct computation[7] that the resulting A_L is given within a neighborhood of threshold by

$$A_L(s)\cong 0.0030(s-4), \quad (7)$$

with $A^{(1)1}$ normalized such that elastic unitarity would imply

$$A^{(1)1}(s)=(1-4/s)^{-1/2}\exp(i\delta)\sin\delta .$$

The above result for $A_L(s)$ might be regarded as small,[8] but it is not zero. Whether it is too small to generate a ρ must be tested by computations. The N/D calculation in Ref. 2 is based in part on the Veneziano A_L, with negative results. The N/D calculation of Ref. 3 is based on a more rigorous model for the left cut, and includes inelasticity, but again fails to generate a ρ. Hence it appears unlikely that the ρ is generated by forces in the $\pi\pi$ channel.

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[1]Gordon L. Shaw, Phys. Rev. D 7, 2265 (1973).

[2]E. P. Tryon, Phys. Lett. 38B, 527 (1972).

[3]E. P. Tryon, Phys. Rev. D 12, 759 (1975).

[4]Cf. C. Lovelace, Phys. Lett. 28B, 264 (1968).

[5]Cf. D. Sivers and J. Yellin, Ann. Phys. (N.Y.) 55, 107 (1969).

[6]It has been shown by R. T. Park and B. R. Desai [Phys. Rev. D 2, 786 (1970)] that the Veneziano $\mathrm{Im}A^{(1)1}$ contains infinitely many oscillations along the left cut, with one oscillation where each tower of resonances is exchanged. As s tends to $-\infty$, the magnitude of $\mathrm{Im}A^{(1)1}$ tends to zero.

[7]I follow Lovelace (Ref. 4) in choosing the trajectory parameters $a=0.483$, $b=0.017$ (with $m_\pi=1$). The overall coefficient β is fixed by the width of the ρ resonance, i.e., I require the ρ pole in $A^{(1)1}$ to correspond to $\mathrm{Im}A^{(1)1}=\pi m_\rho\Gamma_\rho\delta(s-s_\rho)$, with $\Gamma_\rho=1.06$ (i.e., 146 MeV). This leads to $\beta=3m_\rho\Gamma_\rho/(s_\rho-4)=0.66$. The Veneziano $A^{(1)1}$ satisfies the dispersion relation which defines A_L and A_R, with rapidly convergent integrals (cf. Park and Desai, Ref. 6). I compute $A^{(1)1}$ near $s=4$ by a numerical projection of the P wave from A^1, and I compute A_R by integrating over the first 50 resonances in $A^{(1)1}$ (their widths are readily computed from numerical projections of $A^{(1)1}$ out of A^1). A_L is then given by $A_L=A^{(1)1}-A_R$, with the result described by Eq. (7).

[8]The linear approximation (7) yields $A_L=0.05$, 0.09, and 0.15 for $s^{1/2}=0.6$, 0.8, and 1.0 GeV, respectively. The exact Veneziano A_L grows somewhat less rapidly over this range of energy.